MATH 303 – Measures and Integration
Homework 7

Please upload a pdf of your solutions by 23:59 on Monday, November 11. The assignment will be graded out of 16 points (8 for each problem). One problem will be checked for completeness and the other will be graded on correctness and quality. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Problem 1. Let μ be a Lebesgue–Stieltjes measure on $\mathbb{R}$ with $\mu(\mathbb{R}) < \infty$. Let μ^* be the associated outer measure $\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu((a_n, b_n]) : E \subseteq \bigcup_{n\in\mathbb{N}} (a_n, b_n]\right\}$. Define the *inner measure* of a set $E \subseteq \mathbb{R}$ by $\mu_*(E) = \mu(\mathbb{R}) - \mu^*(\mathbb{R} \setminus E)$.

(a) Show that $\mu_*(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\}$.

(b) Prove that a set $E \subseteq \mathbb{R}$ is μ-measurable if and only if $\mu^*(E) = \mu_*(E)$.

Solution: (a) If $K \subseteq E$ is compact, then

$$\mu_*(E) = \mu(\mathbb{R}) - \mu^*(\underbrace{\mathbb{R} \setminus E}_{\subseteq \mathbb{R}\setminus K}) \geq \mu(\mathbb{R}) - \mu^*(\mathbb{R} \setminus K) = \mu(K),$$

since K is μ-measurable. Let us show

$$\mu_*(E) \leq \sup\{\mu(K) : K \subseteq E \text{ compact}\}. \tag{1}$$

Let $\varepsilon > 0$. As demonstrated in the lecture notes (see the proof of Proposition 5.31(2)), we have $\mu^*(\mathbb{R} \setminus E) = \inf\{\mu(U) : U \supseteq \mathbb{R} \setminus E \text{ is open}\}$. Let $U \subseteq \mathbb{R}$ be an open set with $\mathbb{R} \setminus E \subseteq U$ and $\mu(U) < \mu^*(\mathbb{R} \setminus E) + \frac{\varepsilon}{2}$. Let $F = \mathbb{R} \setminus U \subseteq E$, and for $n \in \mathbb{N}$, let $K_n = F \cap [-n, n]$. Then $K_n \subseteq E$ is compact for each $n \in \mathbb{N}$, and

$$\mu(K_n) \to \mu(F) = \mu(\mathbb{R} \setminus U) = \mu(\mathbb{R}) - \mu(U) > \mu(\mathbb{R}) - \mu^*(\mathbb{R} \setminus E) - \frac{\varepsilon}{2} = \mu_*(E) - \frac{\varepsilon}{2}.$$

Taking n sufficiently large so that $\mu(K_n) > \mu(F) - \frac{\varepsilon}{2}$, we then have $\mu(K_n) > \mu_*(E) - \varepsilon$. Letting $\varepsilon \to 0$ establishes the inequality (1).

(b) Suppose E is μ-measurable. Then by parts (2) and (3) of Proposition 5.31,

$$\mu^*(E) = \inf\{\mu(U) : U \supseteq E \text{ is open}\} = \mu(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\} = \mu_*(E).$$

(Alternatively, directly from the definition of measurability, $\mu(\mathbb{R}) = \mu^*(E) + \mu^*(\mathbb{R} \setminus E) = \mu^*(E) + \mu(\mathbb{R}) - \mu_*(E)$, so $\mu^*(E) = \mu_*(E)$.)

Conversely, suppose $\mu^*(E) = \mu_*(E)$. Let $\varepsilon > 0$. Since $\mu^*(E) = \inf\{\mu(U) : U \supseteq E \text{ open}\}$, we may pick an open set $G \subseteq \mathbb{R}$ such that $E \subseteq G$ and $\mu(G) < \mu^*(E) + \frac{\varepsilon}{2}$. Similarly, since $\mu_*(E) = \sup\{\mu(K) : K \subseteq E \text{ compact}\}$, we may pick a compact (hence closed) set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ and $\mu(F) > \mu_*(E) - \frac{\varepsilon}{2}$. Therefore,

$$\mu(G \setminus F) = \mu(G) - \mu(F) < \left(\mu^*(E) + \frac{\varepsilon}{2}\right) - \left(\mu_*(E) - \frac{\varepsilon}{2}\right) = \varepsilon.$$

By Proposition 5.31, it follows that E is μ-measurable, since it satisfies criterion (ii) in part (1).

Problem 2. Let X be an LCH space, and let $\mu : \mathrm{Borel}(X) \to [0, \infty]$ be a Borel measure on X. Prove the following are equivalent:

(i) $\mu(K) < \infty$ for all $K \subseteq X$ compact;

(ii) for every $x \in X$, there exists an open set $U \subseteq X$ with $x \in U$ such that $\mu(U) < \infty$.

Remark: The equivalence between (i) and (ii) may fail if the space is not locally compact. For example, the counting measure on the set of rational numbers $\mathbb{Q}$ satisfies (i) but not (ii). (You do not need to prove this but may want to think about why it is true to deepen your own understanding.)

Solution: (i) $\implies$ (ii). Suppose (i) holds, and let $x \in X$. Since X is locally compact, there is an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that $x \in U \subseteq K$. Condition (i) implies that $\mu(K) < \infty$, so by monotonicity of μ, we have $\mu(U) < \infty$.

(ii) $\implies$ (i). Let K be a compact set. For each $x \in K$, apply property (ii) to find an open neighborhood U_x of x such that $\mu(U_x) < \infty$. By compactness of K, there is a finite subcover; that is, there exists $n \in \mathbb{N}$ and points $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{j=1}^n U_{x_j}$. Therefore, $\mu(K) \leq \sum_{j=1}^n \mu(U_{x_j}) < \infty$.